**フィボナッチ数列の漸化式：特性方程式と一般解**

Recurrence relation

In mathematics, a recurrence relation is an equation that recursively defines a sequence: each term of the sequence is defined as a function of the preceding terms.

A difference equation is a specific type of recurrence relation.

漸化式の例：フィボナッチ数列

Example: Fibonacci numbers

Fn+2 = Fn+1 + fn with initial values F0=0,F1=0.

We obtain the sequence of Fibonacci numbers which begins:

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ...

一般解の導出 The Fibonacci recursion

Fn+2 - Fn+1 - Fn =0

is similar to the defining equation of the golden ratio in the form

x^2 - X -1 = 0

which is also known as the generating polynomial of the recursion. By definition α,β is a root of the equation

α = (1+√5)/2,　β=(1-√5)/2.

Any root of the equation above satisfies x^2 - X -1 = 0 and multiplying by x^n　shows:

x^(n+2)-x^(n+1)-x^n = 0

Both α^n and β^n are geometric series (for n = 1, 2, 3, ...) that satisfy the Fibonacci recursion. Linear combinations of series α^n and β^n, with coefficients a and b, can be defined by

Fc(n)=a・α^n + b・β^n　for any real a and b.

All thus-defined series satisfy the Fibonacci recursion

Fc(n+2)=a・α^(n+2) + b・β^(n+2)

      =a・(α^(n+1)+α^n) + b・(β^(n+1)+β^n)

      =a・α^(n+1)+ b・β^(n+1) + a・α^n+b・β^n

      =Fc(n+1) + Fc(n).

Requiring that Fc(0)=0,Fc(1)=1 yields

Fc(0)=a・α^0 + b・β^0 = a + b =0

Fc(1)=a・α^1 + b・β^1 = a・α + b・β =0

Then　a=1/√5　and　b=-1/√5. The solution of the Fibonacci recursion is

Fn=(α^n - β^n)/√5　=｛(1+√5)/2｝^n　＋　｛(1+√5)/2｝^n

α = (1+√5)/2,　β=(1-√5)/2.

• このことから　特性方程式の解の線形結合で、一般解が求められることが理解できる。

線形差分方程式の特性関数を用いた一般解法

An order linear homogeneous recurrence relation with constant coefficients is an equation of the form:

Fn = a1・Fn-1 + a2・Fn-2 + a3・Fn-3 + ・・・・+am・Fn-m

The characteristic polynomial is

p(x)=x^m-a1・xm-1 + a2・xm-2 + a3・xm-3 + ・・・・+am=0.

For order 1 no theory is needed; the recurrence

Fn=r・Fn-1　with initial condition F0=k.

Note that the characteristic polynomial is simply x-r=0. The most general solution is

Fn=k・r^n　.

Consider, for example, a recurrence relation of the form

Fn=a1・Fn-1 + a2・Fn-2

The characteristic polynomial is

p(x)=x^2-a1・x - a2=0

Solve for x to obtain the two roots λ1, λ2. If these roots are distinct, we have the general solution

Fn = A・λ1^n + B・λ2^n

while if they are identical (when a1^2 + 4a・2 = 0), we have

Fn = A+ B・n・λ2^n

This is the most general solution, the two constants A and B can be chosen freely to produce a solution. If "initial conditions" have been given then we can solve (uniquely) for A and B.

m次元の線形は、特性方程式のm個の解と初期条件を使って、n番目のFnを解のn乗線形結合で表わされることが理解できる。

このことは、線形差分方程式の解が明示的に求められることを意味する。言い換えれば、積分計算をしなくても解析解を示せることを意味する。

Fibonacci sequence