複素空間上の円や螺旋：複素数のべき乗

１．複素数の積とドモアブルの定理

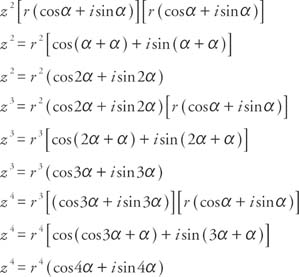
Products of Complex Numbers and De Moivre's Theorem

From Wikipedia, the free encyclopedia

In mathematics, de Moivre's formula (De Moivre's theorem and De Moivre's identity), named after Abraham de Moivre, states that for any complex number (and, in particular, for any real number) x and integer n it holds that

(cos x + i sin x)^n = cos (nx) + i sin (nx).

The process of mathematical induction can be used to prove a very important theorem in mathematics known as De Moivre's theorem. If the complex number z = r(cos α + i sin α), then



**円分多項式　cyclotomic polynomial　：複素空間上のｎ角形**

円周等分多項式或いは円分多項式と呼ばれる次の方程式、  
x^n - 1 = 0  
は、任意の n について、ベキ根と四則演算だけで解を求めることができる。  
  
n=2 の場合=🡺 根の公式  
n=3 の場合=🡺根の公式  
n=4　の場合=🡺明らか  
n=5 の場合=🡺１の５乗根。黄金数が出てくる。  
n=7 の場合=🡺３次方程式 カルダノの公式  
n=17 の場合=🡺２次方程式だけで解ける  
計算方法は大変

x^5=1を解け  
**因数分解による方法**因数分解して  
  
(x-1)(x^4+x^3+x^2+x+1)=0  
左辺よりx=1  
右のカッコ内をx^2≠0に注意してx^2で割ると

x^2+x+1+1/x+1/x^2=0となる。  
これは相反方程式なので整理すると{(x+1/x)^2}-2+(x+1/x)+1=0  
x^2+x=tとして整理するとt^2+t-1=0  
因数分解してt=(-1±√5)/2となる。  
x+1/x=tに戻して  
x^2-((-1±√5)/2)x+1=0  
  
2x^2-(-1±√5)x+2=0  
再び解の公式に入れて x=[(-1+√5)±√(-10-2√5)]/4, [(-1-√5)±√(-10+2√5)]/4.  
で, 2√5 < 10 だから, 虚数単位 i を用いると  
x=[(-1±√5)±i√(10±2√5)]/4  
(√5 の直前の複号だけ同順) となる。  
  
**ドモアブルの公式を使う方法**x=cosθ+i・sinθ とおく。  
x^5=cos5θ+i・sin5θとなる。  
cos5θ=1より、5θ=0+2nπ(nは整数)となる。θ=2nπ/1　n=1,2,3,4.5　の場合が解。  
  
**ド・モアブルの公式**

整数nに対して、  
(cosθ + isinθ)^n = cosnθ + i・sinθ   
が成り立つという複素数に関する定理である  
アブラーム・ド・モアブル（Abraham de Moivre, 1667 - 1754年）はフランスの数学者である。

２．指数関数の４つの定義

The four most common definitions of the exponential function exp(*x*) = *ex* for real *x* are:

1. Define *ex* by the [limit](http://en.wikipedia.org/wiki/Limit_(mathematics))

e^x = \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n.

2. Define *ex* as the value of the [infinite series](http://en.wikipedia.org/wiki/Infinite_series)

e^x = \sum_{n=0}^\infty {x^n \over n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots

(Here *n*! denotes the [factorial](http://en.wikipedia.org/wiki/Factorial) of *n*. One [proof that *e* is irrational](http://en.wikipedia.org/wiki/Proof_that_e_is_irrational) uses this representation.)

3. Define *ex* to be the unique number *y* > 0 such that

\int_{1}^{y} \frac{dt}{t} = x.

This is as the inverse of the [natural logarithm](http://en.wikipedia.org/wiki/Natural_logarithm) function, which is defined by this integral.

4. Define *ex* to be the unique solution to the [initial value problem](http://en.wikipedia.org/wiki/Initial_value_problem)

y'=y,\quad y(0)=1.

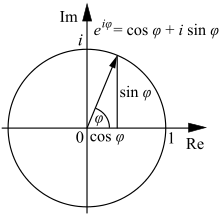
(Here, *y*′ denotes the [derivative](http://en.wikipedia.org/wiki/Derivative) of *y*.)

**３．指数関数の複素領域への拡張：オイラーの恒等式**

[指数関数](http://ja.wikipedia.org/wiki/%E6%8C%87%E6%95%B0%E9%96%A2%E6%95%B0)と[三角関数](http://ja.wikipedia.org/wiki/%E4%B8%89%E8%A7%92%E9%96%A2%E6%95%B0)の間に成り立つ等式

e^{i\theta} =\cos\theta +i\sin\theta

オイラーの公式の幾何的な表示をいう。ここに、*θ* は幾何学的には[弧度法](http://ja.wikipedia.org/wiki/%E3%83%A9%E3%82%B8%E3%82%A2%E3%83%B3" \o "ラジアン)に従う角と見なされる[実変数](http://ja.wikipedia.org/wiki/%E5%AE%9F%E6%95%B0)である

[](http://ja.wikipedia.org/wiki/%E3%83%95%E3%82%A1%E3%82%A4%E3%83%AB:Euler's_formula.svg)

The exponential function *ex* for real values of *x* may be defined in a few different equivalent ways . Several of these methods may be directly extended to give definitions of *ez* for complex values of *z*simply by substituting *z* in place of *x* and using the complex algebraic operations. In particular we may use either of the two following definitions which are equivalent.

**Power series definition**

For complex *z*

e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}.

Using the [ratio test](http://en.wikipedia.org/wiki/Ratio_test) it is possible to show that this [power series](http://en.wikipedia.org/wiki/Power_series) has an infinite [radius of convergence](http://en.wikipedia.org/wiki/Radius_of_convergence), and so defines *ez* for all complex *z*.

**Limit definition**

For complex *z*

e^z = \lim_{n \rightarrow \infty} \left(1+\frac{z}{n}\right)^n ~.

3通りの証明：Proofs

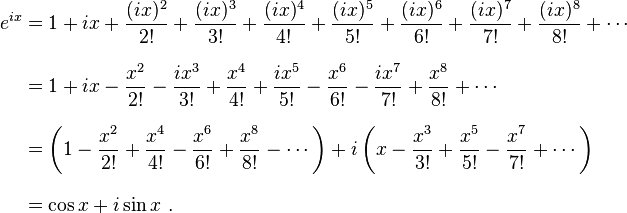
Various proofs of the formula are possible.

**１）Using power series**

Here is a proof of Euler's formula using [power series expansions](http://en.wikipedia.org/wiki/Taylor_series) as well as basic facts about the powers of *i*:

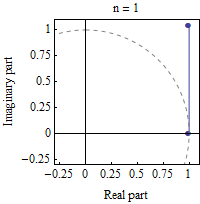
\begin{align} i^0 &{}= 1, \quad & i^1 &{}= i, \quad & i^2 &{}= -1, \quad & i^3 &{}= -i, \ i^4 &={} 1, \quad & i^5 &={} i, \quad & i^6 &{}= -1, \quad & i^7 &{}= -i, \end{align}

and so on. Using now the power series definition from above we see that for real values of *x*



In the last step we have simply recognized the [Maclaurin series](http://en.wikipedia.org/wiki/Taylor_series" \o "Taylor series) for *cos(x)* and *sin(x)*. The rearrangement of terms is justified because each series is [absolutely convergent](http://en.wikipedia.org/wiki/Absolute_convergence).

**２）Using the limit definition**

[](http://en.wikipedia.org/wiki/File:EulerFormulaAsLimit.gif)

[C:\Users\User\AppData\Local\Temp\enhtmlclip\Image(4).png](http://en.wikipedia.org/wiki/File:EulerFormulaAsLimit.gif)

The [exponential function](http://en.wikipedia.org/wiki/Exponential_function) *ez* can be defined as the [limit](http://en.wikipedia.org/wiki/Limit_of_a_sequence) of(1 + *z*/*n*)*n*, as *n* approaches infinity. In this animation, *z*=*iπ*/3, and *n* takes various increasing values from 1 to 100. The computation of (1 + *z*/*n*)*n*is displayed as the combined effect of*n* repeated multiplications in the[complex plane](http://en.wikipedia.org/wiki/Complex_plane). As *n* gets larger, the points approach the complex [unit circle](http://en.wikipedia.org/wiki/Unit_circle) (dashed line), covering an angle of *π*/3 radians.

An alternative proof　is based on the limit definition of e^z:

e^z = \lim_{n\rightarrow\infty} \left(1+\frac{z}{n}\right)^n.

Substitute z=ix, and let *n* be a very large integer, say 1000. Then, based on the limit definition, the complex number (1+*ix*/1000)1000 is supposed to be a good approximation to *eix*. So, what is the value of (1+*ix*/1000)1000?

Consider the sequence of 1000 complex numbers:

1, \, \left(1+\frac{ix}{1000}\right), \, \left(1+\frac{ix}{1000}\right)^2, \ldots, \, \left(1+\frac{ix}{1000}\right)^{1000}

(We started with 1, and successively multiplied it by (1+*ix*/1000), 1000 times.) If the points of this sequence are plotted in the [complex plane](http://en.wikipedia.org/wiki/Complex_plane) (see animation at right), they approximately trace out the [unit circle](http://en.wikipedia.org/wiki/Unit_circle), with each point being *x*/1000 radians counterclockwise of the previous point. (The proof of this is based on the rules of trigonometry and complex-number algebra.) Therefore, the last point in the sequence,(1 + *ix*/1000)1000, is approximately the point on the [unit circle](http://en.wikipedia.org/wiki/Unit_circle) of the [complex plane](http://en.wikipedia.org/wiki/Complex_plane)located *x* radians counterclockwise from +1, that is the point cos *x* + *i* sin *x*. If we replaced the number 1000 by larger and larger numbers, all of the approximations in this paragraph become more and more accurate. Therefore, *eix* = cos *x* + *i* sin *x*.

**３）Using calculus**：微分を用いた証明

Another proof[[7]](http://en.wikipedia.org/wiki/%23cite_note-Strang-7) is based on the fact that all complex numbers can be expressed in polar coordinates. Therefore for some rand \theta depending on x,

e^{ix} = r (\cos(\theta) + i \sin(\theta))\,.

Now from any of the definitions of the exponential function it can be shown that the derivative of e ^{ix} is i e ^{ix}. Therefore differentiating both sides gives

i e ^{ix} = (\cos(\theta) + i \sin(\theta)) \frac{dr}{dx} + r (-\sin(\theta) + i \cos(\theta)) \frac{d \theta}{dx}\,.

Substituting r (\cos(\theta) + i \sin(\theta)) for  and equating real and imaginary parts in this formula gives \textstyle \frac{dr}{dx} = 0 and \textstyle \frac{d\theta}{dx} = 1. Together with the initial values r(0) = 1 and \theta(0) = 0 which come from e^{i0} = 1 this gives r=1 and \theta=x. This proves the formula e^{ix} = 1(\cos(x)+i \sin(x)).

**三角関数の指数関数表現**

この公式は、全く起源の異なる[指数関数](http://ja.wikipedia.org/wiki/%E6%8C%87%E6%95%B0%E9%96%A2%E6%95%B0" \o "指数関数)と[三角関数](http://ja.wikipedia.org/wiki/%E4%B8%89%E8%A7%92%E9%96%A2%E6%95%B0)が[複素数](http://ja.wikipedia.org/wiki/%E8%A4%87%E7%B4%A0%E6%95%B0)の世界では密接に結びついていることを示していると見ることができる。たとえば三角関数の[加法定理](http://ja.wikipedia.org/wiki/%E4%B8%89%E8%A7%92%E9%96%A2%E6%95%B0#.E4.B8.89.E8.A7.92.E9.96.A2.E6.95.B0.E3.81.AE.E5.8A.A0.E6.B3.95.E5.AE.9A.E7.90.86)は、指数法則 *eaeb* = *ea+b* に対応していることが分かる。さらに

\cos z=\frac{e^{iz}+e^{-iz}}{2},

\sin z =\frac{e^{iz}-e^{-iz}}{2i}

と置き換えることで、[初等関数](http://ja.wikipedia.org/wiki/%E5%88%9D%E7%AD%89%E9%96%A2%E6%95%B0)は全て指数関数の一部であると見なすこともできる。

**初等関数**とは、[複素数](http://ja.wikipedia.org/wiki/%E8%A4%87%E7%B4%A0%E6%95%B0)を[変数](http://ja.wikipedia.org/wiki/%E5%A4%89%E6%95%B0_(%E6%95%B0%E5%AD%A6))とする[多項式関数](http://ja.wikipedia.org/wiki/%E5%A4%9A%E9%A0%85%E5%BC%8F%E9%96%A2%E6%95%B0)・[指数関数](http://ja.wikipedia.org/wiki/%E6%8C%87%E6%95%B0%E9%96%A2%E6%95%B0)・[対数関数](http://ja.wikipedia.org/wiki/%E5%AF%BE%E6%95%B0%E9%96%A2%E6%95%B0)主値の[四則演算](http://ja.wikipedia.org/wiki/%E5%9B%9B%E5%89%87%E6%BC%94%E7%AE%97)・[合成](http://ja.wikipedia.org/wiki/%E5%86%99%E5%83%8F#.E5.86.99.E5.83.8F.E3.81.AE.E5.90.88.E6.88.90)によって表示できる[関数](http://ja.wikipedia.org/wiki/%E9%96%A2%E6%95%B0_(%E6%95%B0%E5%AD%A6))である。これによると、[三角関数](http://ja.wikipedia.org/wiki/%E4%B8%89%E8%A7%92%E9%96%A2%E6%95%B0)や[双曲線関数](http://ja.wikipedia.org/wiki/%E5%8F%8C%E6%9B%B2%E7%B7%9A%E9%96%A2%E6%95%B0)、そして両者の[逆関数](http://ja.wikipedia.org/wiki/%E9%80%86%E9%96%A2%E6%95%B0)[主値](http://ja.wikipedia.org/wiki/%E4%B8%BB%E5%80%A4)も

\sin x = \frac{\exp(ix) - \exp(-ix)}{2i}

\sinh x = \frac{\exp(x) - \exp(-x)}{2}

\operatorname{arcsin}x = -i\operatorname{ln}(ix + \sqrt{1 - x^{2}})

\begin{align} \operatorname{sinh}^{-1}x &= \operatorname{Areasinh}x \\ &= \operatorname{ln}(x + \sqrt{1 + x^{2}}) \end{align}

に代表される表示が可能であるから、初等関数と考えることが出来る。

**初等関数の**[**導関数**](http://ja.wikipedia.org/wiki/%E5%B0%8E%E9%96%A2%E6%95%B0)**は必ず初等関数になる.**

1. **連続回微分可能な初等関数は、ある点の周りで多項式で近似できる。このような多項式展開を、マクローリン展開とかテーラー展開と呼ばれる。**

* 三角関数と指数関数は，次のようにテイラー展開できる．

|  |  |  |
| --- | --- | --- |
| $\displaystyle \cos x$ | $\displaystyle =1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!} +\frac{x^8}{8!}-\... ...{x^{12}}{12!} -\frac{x^{14}}{14!}+\frac{x^{16}}{16!}-\frac{x^{18}}{18!} +\cdots$ |  |
| $\displaystyle \sin x$ | $\displaystyle =x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!} +\frac{x^9}{9!}-\... ...{x^{13}}{13!} -\frac{x^{15}}{15!}+\frac{x^{17}}{17!}-\frac{x^{19}}{19!} +\cdots$ |  |
| $\displaystyle e^x$ | $\displaystyle =1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!} +\frac{x^5}{5!}... ...^6}{6!}+\frac{x^7}{7!}+\frac{x^8}{8!} +\frac{x^9}{9!}+\frac{x^{10}}{10!}+\cdots$ |  |

ある関数が、下記の式で表すことができるとしよう

|  |  |  |
| --- | --- | --- |
| $\displaystyle f(x)$ | $\displaystyle =a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+\cdots$$\displaystyle =\sum_{n=0}^\infty a_nx^n$ |  |

．$ f(x)$は三角関数であったり，指数関数，あるいは対数関数である．この式の左辺は，冪級数と呼ぶ．この式は，任意の関数を冪球数に展開して いるのである．

任意の関数$ f(x)$が式のように冪級数に展開できるならば，そ の係数$ a_n$をどうやって決めるのか?元の式，お よびその両辺を繰り返し微分すると次の式が得られる．

|  |  |
| --- | --- |
| $\displaystyle f(x)$ | $\displaystyle =a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+a_6x^6+\cdots$ |
| $\displaystyle f^\prime(x)$ | $\displaystyle =a_1+2a_2x+3a_3x^2+4a_4x^3+5a_5x^4+6a_6x^5\cdots$ |
| $\displaystyle f^{\prime\prime}(x)$ | $\displaystyle =2\times1\times a_2+3\times2a_3x+4\times3a_4x^2 +5\times4a_5x^3+ 6\times5a_6x^4\cdots$ |
| $\displaystyle f^{(3)}(x)$ | $\displaystyle =3\times2\times1\times a_3+4\times3\times2a_4x+5\times4\times3a_5x^2+ 6\times5\times4a_6x^3\cdots$ |
|  | $\displaystyle \qquad\vdots$ |
| $\displaystyle f^{(n)}(x)$ | $\displaystyle =n!a_n+\frac{(n+1)!}{1!}a_{n+1}x+\frac{(n+2)!}{2!}a_{n+2}x^2 +\frac{(n+3)!}{3!}a_{n+3}x^3+\cdots$ |

これらの式で$ x=0$とすると，右辺第2項より高次の項は全てゼロとなる．これを利用して， 式([7](http://akita-nct.jp/yamamoto/lecture/2006/3E/2nd/html/node2.html#eq:power_series))の展開係数は

|  |  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | $\displaystyle a_0=f(0)$ |  | $\displaystyle a_1=f^\prime(0)$ |  | $\displaystyle a_2=\frac{f^{\prime\prime}(0)}{2\times1}$ |  | $\displaystyle a_3=\frac{f^{(3)}(0)}{3\times2\times1}$ |  | $\displaystyle \cdots$ |  | $\displaystyle a_n=\frac{f^{(n)}(0)}{n!}$ |  |  |

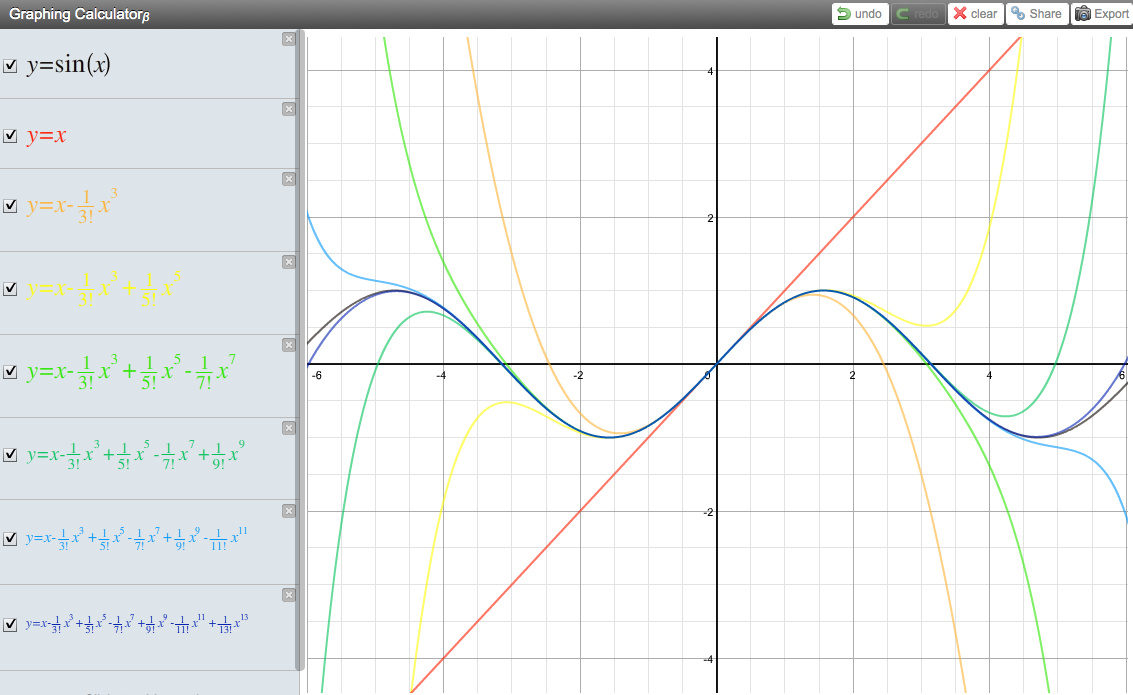
と得られる．$ 0!=1$とすると，最後の式である $ a_n=f^{(n)}(0)/n!$が$ n=0$を含めて一般的 に成り立つ．整理すれば

|  |  |  |
| --- | --- | --- |
| $\displaystyle f(x)$ | $\displaystyle =f(0)+f^\prime(0)x+\frac{f^{\prime\prime}(0)}{2!}x^2+\frac{f^{(3)... ...}x^3 +\frac{f^{(4)}(0)}{4!}x^4+\cdots=\sum_{n=0}^\infty\frac{f^{(n)}(0)}{n!}x^n$ |  |

となる．これで，任意の関数を冪級数で展開する方法が分かった

任意の関数は，冪級数に展開できる． マクローリン展開はゼロの周りで近似。

|  |
| --- |
| $\displaystyle f(x)=f(0)+f^\prime(0)x+\frac{f^{\prime\prime}(0)}{2!}x^2 +\frac{f... ...}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots =\sum_{n=0}^\infty\frac{f^{(n)}(0)}{n!}x^n$ |

**三角関数の原点回りでの多項式近似**

**付録：Appendix**

**Proof of e^(ix) = cos(x) + isin(x)**

**From: Walter Graf**

**Subject: Proof of e^(ix) = cos(x) + isin(x)**

**In the equation e^(iPi) - 1 = 0, the proof is to evaluate**

**e^(ix) = cos(x) + isin(x) for x = Pi.**

**I would like to see a rigorous proof of of the the above equation.　　　　Thank you,**

**From: Doctor Mitteldorf**

**Subject: Re: Proof of e^(ix) = cos(x) + isin(x)**

**Dear Walter,**

**This is called the Euler equation, and it's not something you can prove rigorously. It's a definition, and I'd like to convince you that it's the only sensible definition, of how to compute imaginary exponentials.**

**I can think of three approaches to verifying the Euler equation, but unfortunately one of them is all calculus, one uses calculus explicitly, and only the third is free of calculus. I'm just guessing from your age that you may not have studied calculus yet.**

**You can verify that the Euler equation makes a sensible definition by expanding the two sides as Taylor series in x. You can also differentiate both sides and see that the answer is self-consistent.**

**Thirdly, you can use the formula for cos(2x) and sin(2x) to show that the right side has the property you expect from an exponential, so**

**that e^i(2x) = (e^ix)^2.**

**So start with choice 3. You have the formulas**

**cos(2x) = cos^2(x) - sin^2(x) and**

**sin(2x) = 2 sin(x) cos(x)**

**You'd also want to demand that e^i(2x) = (e^ix)^2. That means that your new definition of e^ix is behaving like an exponential. See if you can put these together to show that**

**e^i(2x) = cos (2x) + i sin(2x).**

**The Taylor expansion is something you can appreciate without calculus, although its roots are in calculus. It's a series expression for a function. You may have run across the following infinite series representations of cos and sin and e^x. In fact, this is the most straightforward way to compute the value of sin(x) or e^x for any**

**given x.**

**cos(x) = 1 - x^2/2! + x^4/4! - x^6/6! + ...**

**sin(x) = x - x^3/3! + x^5/5! - x^7/7! + ...**

**e^x = 1 + x + x^2/2! + x^3/3! + x^4/4! + ...**

**The ! in these equations means factorial. In other words,**

**4! = 4\*3\*2\*1.**

**See if you can use these infinite series expressions to verify the Euler equation.**

**-Doctor Mitteldorf, The Math Forum**

**Check out our web site! http://mathforum.org/dr.math/**

**Euler Equation and DeMoivre's Theorem**